Accelerated FD Analysis of Dielectric Resonators

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***Abstract*—Finite-difference frequency-domain (FDFD) formulation, based on Yee's mesh, is used for determination of resonant frequencies of rotationally symmetric shielded dielectric resonators. Resulting eigenproblems are solved by means of the Arnoldi method with Chebyshev preconditioning.**

***Index Terms*— Arnoldi method, dielectric resonators, finite-difference methods.**

I. Introduction

Shielded dielectric resonators (DR's) using anisotropic uniaxial crystals, such as sapphire, as well as isotropic media are widely used in microwave technique due to their very low-loss nature. Numerical techniques for the analysis of such DR's include modal expansions with the Rayleigh–Ritz method [1], mode matching [2], [3], finite-element method (FEM) [1], or finite-difference frequency domain (FDFD) based on Yee's grid [4] or on a single collocated mesh [5], [6]. Amongst them, the most versatile and able to deal with arbitrary shaped boundaries and inlays are the FDFD and FEM methods. Depending on analytical formulation used for deriving numerical equations, these techniques transform an infinite dimensional electromagnetic eigenvalue problem into a large sparse symmetric or nonsymmetric matrix eigenvalue problem. Compared to FEM, the FDFD results in a standard, rather than a generalized, eigenproblem. Standard problems are easier to deal with using numerical techniques and their solution is faster. Numerical algorithms which are recommended for finding a few eigenvalues of a large nonsymmetric eigenproblem include the subspace iteration or simultaneous iteration (SI) method and the Arnoldi method [7]. These techniques can be substantially improved using polynomial preconditioning, i.e., of the Chebyshev type [5], [7]. The aim of this letter is to compare accuracy of two FDFD formulations and to examine performance of the algorithm based on the Arnoldi method with Chebyshev preconditioning relative to accelerated version of SI method (SIC) reported in [5] and [6].

II. Finite-Difference Formulation

Finite differences in electromagnetic problems can be implemented in two ways. One approach is to use Yee's mesh [4] which allows an easy implementation of integral forms of Maxwell's equations. The second approach [5], [6] uses a single collocated mesh and is used mainly to discretize wave equation. The former discretization method has an advantage that additional equations are not required to satisfy the boundary conditions.

In this letter we use the approach based on Yee's mesh similar to the one described in [4]. We consider rotationally symmetric structures so that a three-dimensional (3-D) problem can be converted to an equivalent two-dimensional (2-D) one in a transverse r–z space. Divergence equation is used to eliminate the azimuthal ϕ-component from the full $\boldsymbol{E}$ or $\boldsymbol{H}$ formulation leading to a sparse nonsymmetric eigenproblem for hybrid modes EH_{m*}, HE_{m*} $(m > 0)$. Such an approach reduces the size of the eigenproblems and also avoids spurious zero-eigenvalues. The eigenvectors of the resulting eigenproblem are E_r–E_z or H_r–H_z fields, and corresponding eigenvalues are squares of angular frequencies ω.

For azimuthally invariant modes $(m = 0)$, formulation for one azimuthal ϕ-component is used, which leads to the sparse nonsymmetric eigenproblems for TE_0 and TM_0 modes. The eigenvectors are E_ϕ or H_ϕ fields, respectively, and corresponding eigenvalues are squares of angular frequencies ω.

III. Solution of the Eigenproblem

For the solution of the eigenproblem we used the implicitly restarted Arnoldi algorithm.[1] This method requires calculation of a product of a matrix operator and a vector provided by the algorithm. Each Arnoldi iteration requires calculation of this product at most $p-q$ times (except of the first one which needs p products), where q is the number of eigenvalues to be found and p $(p > q)$ is the size of the subspace. In practice p is chosen to be much smaller than the size n of matrix operator. Internally, each iteration of the Arnoldi algorithm requires the solution of the $p \times p$ eigenproblem by means of the QR method. External calculation of the matrix–vector product and internal solution of the $p \times p$ eigenproblem dominate the total time of the Arnoldi algorithm.

Standard Chebyshev preconditioning technique was implemented in the way similar to the one described in [5]. Chebyshev polynomial of order k is applied to an operator matrix via a recurrence formula so that k matrix–vector products are calculated each time when the product of the operator matrix and the vector is required.

IV. Numerical Results

In order to test the accuracy of the FDFD formulation we calculated resonant frequencies of a cylindrical DR structure presented in Fig. 1, partially filled with dielectric inlays. We

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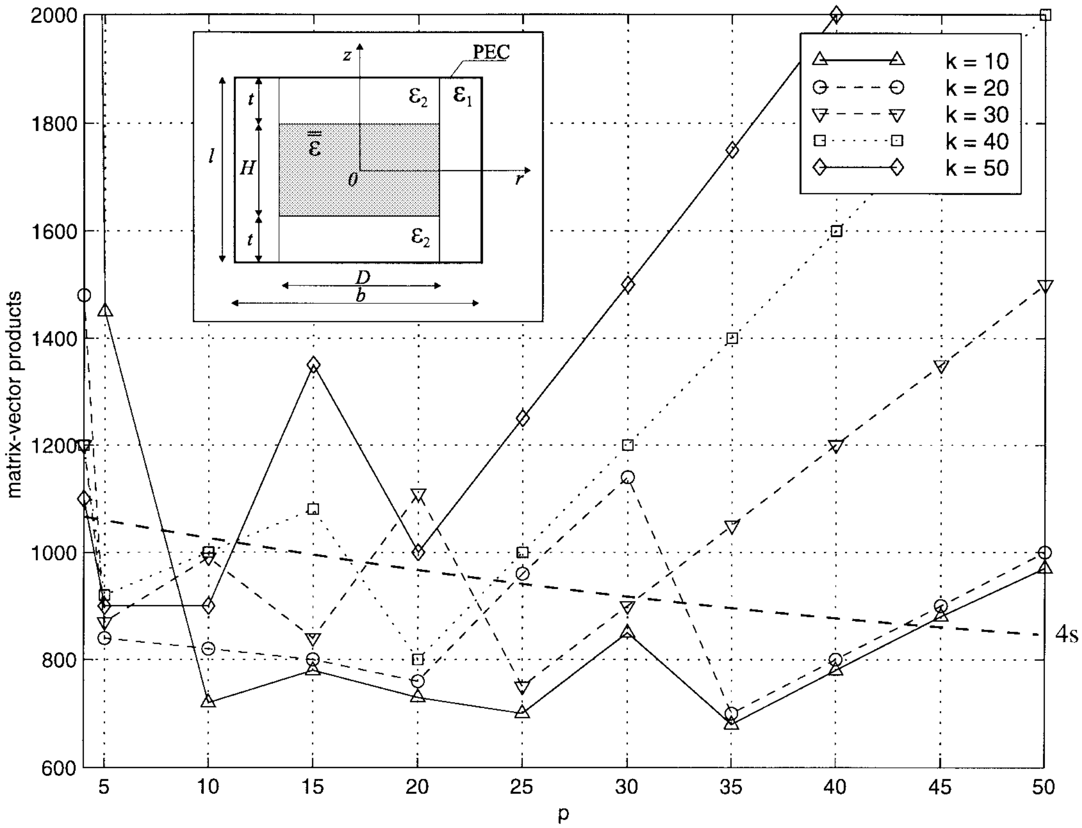

Fig. 1. DR structure and number of matrix–vector products taken in the analysis of the isotropic DR for different values of p and k.

TABLE I
COMPARISON OF RESONANT FREQUENCIES [GIGAHERTZ] FOR THE SAPPHIRE DR (D = 10.0 mm, b = 15.6 mm, l = 13.0 mm, $\epsilon_1 = 1$, $\epsilon_2 = 1.031$)

	$H = 10.0$mm, $\epsilon_r = \epsilon_\phi = 9.389$, $\epsilon_z = 11.478$					
Mode	TM_{01}	EH_{11}	HE_{11}	TE_{01}	HE_{21}	EH_{21}
Present FDFD	7.322	8.822	9.114	9.709	12.053	13.514
FD-SIC [6]	7.359	8.845	9.113	9.706	12.048	13.522
Mode-Matching [3]	7.339	8.827	9.121	9.720		
Difference rel. to [3]	0.23%	0.06%	0.08%	0.11%		
	$H = 5.0$mm, $\epsilon_r = \epsilon_\phi = 9.399$, $\epsilon_z = 11.553$					
Mode	EH_{11}	TM_{01}	TE_{01}	HE_{11}	HE_{21}	EH_{21}
Present FDFD	9.848	10.651	10.704	12.157	14.594	15.244
FD-SIC [6]	9.873	10.696	10.696	12.163	14.595	15.254
Mode Matching [3]	9.841	10.664	10.704	12.153	—	
Difference rel. to [3]	−0.07%	0.12%	0.00%	−0.03%	—	–

used a 78 × 65 equidistant grid taking into account the symmetry in the z-direction (only one half of the structure was analyzed). Two cases with different heights and slightly different permittivities of the sapphire rod were considered. Resonant frequencies of the dominant TE_0 and TM_0 modes ($n \approx 5000$) and hybrid HE and EH modes ($n \approx 10\,000$) were computed. The results of calculations are presented in Table I. For comparison, the results for FDFD based on a single-mesh [6] and mode-matching method [3] are also included. One can see that the resonant frequencies calculated with the FDFD incorporating Yee's grid are more accurate than the ones obtained with a single grid. It means that the implementation of boundary conditions in Yee's formulation is more accurate. The maximum difference relative to the mode matching method does not exceed 0.25%.

TABLE II
COMPARISON OF RESONANT FREQUENCIES [GIGAHERTZ] FOR THE DR LOADED WITH A DIELECTRIC ROD (D = 0.68 in, H = 0.3 in, b = 1.02 in, l = 0.6 in, $\epsilon_1 = 1$, $\epsilon_2 = 1$, $\epsilon_r = \epsilon_\phi = \epsilon_z = 35.74$)

Mode	TE_{01}	EH_{11}	HE_{11}	TM_{01}	HE_{21}	EH_{21}
Present FDFD	3.433	4.229	4.318	4.541	5.000	5.323
FD-SIC [5]	3.429	4.205	4.310	4.542	4.992	5.311
Mode Matching [2]	3.428	4.224	4.326	4.551	5.00	5.33
Difference rel. to [2]	−0.15%	−0.12%	0.18%	0.22%	0.00%	0.13%

We also analyzed a cylindrical cavity loaded with an isotropic dielectric rod. An equidistant 51 × 30 grid was used. The size of the resulting eigenproblem was $n \approx 3000$ for hybrid modes while $n \approx 1500$ for TE_0 and TM_0 modes. Calculated resonant frequencies are presented in Table II along with the results for the FDFD method based on single grid [5] and the mode-matching method [2]. As in the first test, the FDFD with Yee's grid appears to be more accurate (except for TE_{01} and TM_{01} modes) than the single-grid formulation and the maximum difference relative to the mode-matching method was 0.22%.

To test the efficiency of the solver and compare it with SI we computed two dominant ($q = 2$) TE_0 odd modes (magnetic wall in the symmetry plane) for the structure described above, with accuracy to the third decimal. The results are presented in Table III. For unaccelerated case we observed that as the size of subspace p increases, the number of iterations decreases and so does the number of matrix–vector products. Nevertheless, the total calculation time increases due to more costly internal $p \times p$ eigenproblem solutions for larger values of p. On

TABLE III
COMPUTATIONAL COST OF ANALYSIS OF THE ISOTROPIC DR IN UNACCELERATED CASE AND CALCULATION TIME FOR OPTIMAL ORDER OF CHEBYSHEV POLYNOMIAL k

p	4	5	10	15	20	25	30	35	40	50
iterations	5343	1483	155	84	52	37	28	30	20	17
mat-vec	10688	4443	1228	1083	933	850	782	987	760	816
time [s]	106.6	43.2	12.6	12.7	12.8	13.3	13.4	17.9	15.3	19.1
acc. time [s]	3.9	3.2	2.9	3.1	3.1	3.1	3.5	2.9	3.3	4.3
(k_{opt})	(60)	(20)	(10)	(20)	(20)	(10)	(30)	(20)	(20)	(20)

the other hand, when p decreases, the number of required matrix–vector products rapidly grows up and the total time also increases.

These tradeoffs can be easily mitigated when the Arnoldi method is combined with Chebyshev preconditioning technique causing that the number of iterations taken is significantly reduced. For example, for $10 \leq p \leq 20$ the decrease is by the factor of ≈15 for $k = 10$ and ≈30 for $k = 20$. This causes that total calculation time is dominated by the time spent in matrix–vector operations. The number of computed products is shown in Fig. 1 for a few different Chebyshev polynomial orders. We see that there are many cases where we need only 700 products to get convergence, especially for $10 \leq p \leq 35$ and $k = 10$. This result is 30% better than the one reported in [5], where subspace iteration solver needed 1000 products. For large values of k the Arnoldi algorithm makes only one iteration. It requires kp matrix–vector products, and if p in not sufficiently small the calculations are inefficient.

As can be seen in Table III, the optimal total calculation time for accelerated case was 12.6 s for $p = 10$ on our *SGI Power Challenge* architecture. The shortest times are observed for p and k offering the smallest number of matrix–vector products, i.e., 2.9 s for $k = 10$ and p the same as above, which gives the speedup factor >4. All points below a thick dotted line in Fig. 1 indicate the total calculation times smaller than 4 s and correspond to the speedups of the same order. General observations for other mode types (TM, HE, EH) are similar.

V. CONCLUSIONS

The FDFD formulation based on Yee's dual grid was found to be more accurate in implementing boundary conditions than the one based on single grid. Performance of the Arnoldi solver with Chebyshev preconditioning strongly depends on the choice of the subspace size p and the polynomial order k. Once they are properly chosen, implicitly restarted Arnoldi method can offer considerable memory and time savings and can be more efficient than subspace iteration.

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